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Dirichlet–Neumann Bracketing for Horn-Shaped Regions

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We use Dirichlet–Neumann bracketing to obtain sharp upper and lower bounds for the spectral counting function of the Dirichlet laplacian for a horn-shaped region in $\mathbb{R}^m$. The first and second term (and an estimate for the remainder) in the asymptotic expansion of the spectral counting function are obtained for a region in $\mathbb{R}^2$ given by $\{(x_1, x_2) : x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ |x_1| \cdot |x_2|^\alpha < 1\}$, $2^{-1/2} < \alpha < 2^{1/2}$.

1. Introduction

Let D be an open bounded set in $\mathbb{R}^m$ $(m = 2, 3, \ldots,)$ with a piecewise smooth boundary ∂D and let $-\Delta_D$ be the Dirichlet laplacian for D. A theorem of H. Weyl [1] asserts that the spectrum of $-\Delta_D$ is discrete $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ and

$$\lim_{\lambda \to \infty} N_D(\lambda)\, \lambda^{-m/2} = (4\pi)^{-m/2}\, (\Gamma(1 + m/2))^{-1}\, |D|, \tag{1}$$

where $|D|$ is the volume of D and

$$N_D(\lambda) = \#\{j : \lambda_j < \lambda\}. \tag{2}$$

Weyl's theorem has been generalized to the case where D is an open set in $\mathbb{R}^m$ with finite volume [2, 3].

F. Rellich proved in [4] that there exist half tubes in $\mathbb{R}^m$ $(m = 2, 3, \ldots,)$ with infinite volume for which the spectrum of the corresponding Dirichlet laplacian is discrete. We refer to [5, and Refs. therein] for necessary and sufficient conditions for discreteness of the spectrum of the Dirichlet laplacian for general open sets in $\mathbb{R}^m$. While it is possible to obtain upper and lower bounds for the spectral counting function in a variety of

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"non-Weyl" situations [6–10], precise asymptotics are available in a few special cases only.

In this paper we will obtain estimates for the spectral counting function for horn-shaped regions and improve some of the results in [11–13, 15, 16]. The following was proved in [16].

THEOREM 1. *Let* $f_1\colon [0, \infty) \to \mathbb{R}^+$, $f_2\colon [0, \infty) \to \mathbb{R}^+$ *be right continuous and decreasing to* 0. *Let* $D \subseteq \mathbb{R}^2$ *be*

$$D = \{(x_1, x_2)\colon x_1 > 0,\ -f_1(x_1) < x_2 < f_2(x_1)\}, \tag{3}$$

$$f(x) = f_1(x) + f_2(x), \qquad x > 0, \tag{4}$$

and suppose that

$$\int_{[0,\infty)} e^{-tf^{-2}(x)}\, dx < \infty, \qquad t > 0. \tag{5}$$

Then

$$Z_D(t) \sim \frac{1}{(4\pi t)^{1/2}} \int_{[0,\infty)} dx \sum_{k=1}^{\infty} e^{-t\pi^2 k^2 f^{-2}(x)}, \qquad t \downarrow 0, \tag{6}$$

where

$$Z_D(t) = \operatorname{trace}(e^{t\Delta_D}), \tag{7}$$

$(A(t) \sim B(t),\ t \downarrow 0 \Leftrightarrow \lim_{t \to 0} A(t)/B(t) = 1)$.

Since

$$Z_D(t) = \int_{[0,\infty)} e^{-t\lambda}\, dN_D(\lambda), \tag{8}$$

formula (6) suggests that

$$N_D(\lambda) \sim \int_{[0,\infty)} dx \sum_{k=1}^{\infty} \left\{\left\{\frac{\lambda}{\pi^2} - \frac{k^2}{f^2(x)}\right\}^+\right\}^{1/2}, \qquad \lambda \to \infty, \tag{9}$$

$(x^+ = (x + |x|)/2)$. However, the tauberian theorem required [17] assumes growth conditions on $Z_D(t)$ which are not satisfied in general. Furthermore, the right hand side of (9) is always finite even if condtion (5) is not satisfied.

In this paper we resolve these problems. The main result is the following.

THEOREM 2. *Let $f_1: [0, \infty) \to \mathbb{R}^+$, $f_2: [0, \infty) \to \mathbb{R}^+$ be right continuous and decreasing to 0. Let D and f be given by* (3) *and* (4), *respectively. Then for $\lambda > 0$*

$$\left| N_D(\lambda) - \int_{[0,\infty)} dx \sum_{k=1}^{\infty} \left\{ \left\{ \frac{\lambda}{\pi^2} - \frac{k^2}{f^2(x)} \right\}^+ \right\}^{1/2} \right|$$
$$\leqslant \pi^{-1} f(0)\, \lambda^{1/2} + 2\pi^{-3/2} f^{1/2}(0)\, \lambda^{3/4} \left\{ \int_{\{x : f(x) \geqslant \pi\lambda^{-1/2}\}} dx\, f(x) \right\}^{1/2}. \quad (10)$$

Dirichlet–Neumann bracketing has been used in [14] to obtain a sharper form of Weyl's theorem. For D bounded in $\mathbb{R}^m$ $(m = 2, 3)$ with piecewise smooth boundary ∂D one finds by bracketing

$$N_D(\lambda) = (4\pi)^{-m/2} (\Gamma(1 + m/2))^{-1} \, |D| \, \lambda^{m/2} + O(\lambda^{(m-1)/2} \log \lambda), \qquad \lambda \to \infty. \quad (11)$$

However, the sharp remainder is $O(\lambda^{(m-1)/2})$ [18–20]. There are few examples where the second term in the asymptotic expansion of $N_D(\lambda)$ as $\lambda \uparrow \infty$ is known [21, 22]. In Corollary 3 we give such an example. This example has been studied by various authors [11, 13, 15] all of whom obtain the leading term only. Here we use Dirichlet–Neumann bracketing (Theorem 2) to obtain (i) the second term and an estimate for the remainder if $1 \leqslant \alpha < 2^{1/2}$ and (ii) and estimate for the remainder if $\alpha \geqslant 2^{1/2}$.

COROLLARY 3. *Let $D_\alpha \subseteq \mathbb{R}^2$ be given by*

$$D_\alpha = \{(x_1, x_2) : x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ |x_1| \cdot |x_2|^\alpha < 1\}, \quad (12)$$

where x_1 and x_2 are the coordinates with respect to a cartesian frame. Then $N_{D_\alpha}(\lambda) = N_{D_{1/\alpha}}(\lambda)$ and as $\lambda \to \infty$

$$N_{D_1}(\lambda) = \pi^{-1} \lambda \log \lambda + 2\pi^{-1} \lambda (\gamma - \log \pi) + O(\lambda^{3/4} (\log \lambda)^{1/2}), \quad (13)$$

$$N_{D_\alpha}(\lambda) = \beta(\alpha)\, \lambda^{(1+\alpha)/2} + \beta(1/\alpha)\, \lambda^{(1+\alpha)/(2\alpha)} + O(\lambda^{(2+\alpha)/4}), \quad 1 < \alpha < 2^{1/2}, \quad (14)$$

$$N_{D_\alpha}(\lambda) = \beta(\alpha)\, \lambda^{(1+\alpha)/2} + O(\lambda^{(2+\alpha)/4}), \qquad \alpha \geqslant 2^{1/2}, \quad (15)$$

where γ is Euler's constant,

$$\beta(\alpha) = 2^\alpha \pi^{-\alpha - 1/2} \zeta(\alpha) \frac{\Gamma((2+\alpha)/2)}{\Gamma((3+\alpha)/2)}, \quad (16)$$

and $\zeta(\alpha)$ is the Riemann zeta function defined on $\mathbb{C} \setminus \{1\}$.

COROLLARY 4. *Let* $E_\alpha \subseteq \mathbb{R}^2$ *be given by*

$$E_\alpha = \{(x_1, x_2) : x_1 > 0, 0 < x_2 < (\log(e + x_1))^{-\alpha}\}, \tag{17}$$

where $\alpha > 0$. *Then as* $\lambda \to \infty$

$$N_{E_\alpha}(\lambda) \sim e^{(\lambda/\pi^2)^{1/(2\alpha)}} \lambda^{(2\alpha - 1)/(4\alpha)} \pi^{(1-\alpha)/(2\alpha)} (\alpha/2)^{1/2}. \tag{18}$$

Note that condition (5) is not satisfied for $0 < \alpha \leqslant \frac{1}{2}$. For $\alpha > \frac{1}{2}$ condition (5) is satisfied but $Z_{E_\alpha}(t)$ grows exponentially fast as $t \downarrow 0$, so that Karamata's tauberian theorem cannot be applied.

Finally in Corollary 5 we show that Theorem 2 implies Theorem 1 provided (5) holds.

COROLLARY 5. *Suppose* $D \subseteq \mathbb{R}^2$ *satisfies the hypothesis of Theorem* 1. *Then* (10) *implies* (6).

This paper is organized as follows. In Section 2 we use Dirichlet–Neumann bracketing to obtain an estimate for the counting function for horn-shaped regions in $\mathbb{R}^m$ (Theorem 8). We then prove Theorem 2 as a special case.

In Section 3 we will outline the proofs of the Corollaries 5 and 3. We omit the proof of Corollary 4 since it follows from Theorem 2 by a straightforward computation.

2. HORN-SHAPED REGIONS

Let $\{x_1, ..., x_m\}$ be the coordinates of a point $x \in \mathbb{R}^m$ $(m = 2, 3, ...,)$ with respect to a cartesian frame. Let P_ρ be the $(m-1)$-dimensional plane $x_m = \rho$. For any set D in $\mathbb{R}^m$ we define $D(\rho)$ to be the orthogonal projection of $P_\rho \cap D$ onto P_0.

DEFINITION 6. A set $D \subseteq \mathbb{R}^m$ is (one-sided) horn-shaped if (i) D is open and connected, (ii) $D(\rho) \subseteq D(\rho')$ for all $\rho \geqslant \rho' > 0$, (iii) $D(\rho) = \phi$ for $\rho \leqslant 0$.

Part (i) of the following is close to [4, Theorem 1].

THEOREM 7. *Let D be a (one-sided) horn-shaped set in* $\mathbb{R}^m$. *Let* $D_0 = \lim_{\rho \downarrow 0} D(\rho)$ *and suppose that the spectrum of the* $(m-1)$*-dimensional Dirichlet laplacian* $-\Delta_{D_0}$ *is discrete and given by* $\lambda_1(0) < \lambda_2(0) \leqslant \lambda_2(0) \cdots$. *Let* $\lambda_1(\rho) \leqslant \lambda_2(\rho) \leqslant \lambda_3(\rho) \leqslant \cdots$ *be the spectrum of the* $(m-1)$*-dimensional Dirichlet laplacian* $-\Delta_{D(\rho)}$. *Then*

(i) *The spectrum of* $-\Delta_D$ *is discrete if and only if* $\lim_{\rho \to \infty} \lambda_1(\rho) = +\infty$.

(ii) *Suppose* $\lim_{\rho\to\infty}\lambda_1(\rho)=+\infty$. *Then for any* $\delta>0$

$$\sum_{i=1}^{\infty}\#\left\{(k,l):k\in\mathbb{Z}^+,l\in\mathbb{Z}^+,\left(\frac{\pi l}{\delta}\right)^2+\lambda_k(i\delta)<\lambda\right\}\leqslant N_D(\lambda)$$
$$\leqslant\sum_{i=0}^{\infty}\#\left\{(k,l):k\in\mathbb{Z}^+,l\in\mathbb{Z}^+\cup\{0\},\left(\frac{\pi l}{\delta}\right)^2+\lambda_k(i\delta)<\lambda\right\}. \tag{19}$$

Proof. We denote by $[x]$ the integer n such that $n\leqslant x<n+1$. Let $D^+\subseteq\mathbb{R}^m$ be the (one-sided) horn-shaped set defined by

$$D^+(\rho)=\begin{cases}D_0, & 0<\rho<\delta,\\ D([\rho/\delta]\,\delta), & \rho\geqslant\delta,\\ \phi, & \rho\leqslant 0,\end{cases} \tag{20}$$

and let

$$D_0^+=D^+\Big\backslash\bigcup_{n=1}^{\infty}(P_{n\delta}\cap D^+). \tag{21}$$

Consider the Laplace operator with Neumann boundary conditions on $\bigcup_{n=0}^{\infty}(\partial D_0^+\cap P_{n\delta})$ and Dirichlet boundary conditions on $\partial D_0^+\backslash\bigcup_{n=0}^{\infty}(\partial D_0^+\cap P_{n\delta})$. The spectrum of this mixed operator is given by

$$\bigcup_{i=0}^{\infty}\bigcup_{l=0}^{\infty}\bigcup_{k=1}^{\infty}\left\{\left(\frac{\pi l}{\delta}\right)^2+\lambda_k(i\delta)\right\}. \tag{22}$$

Suppose $\lim_{\rho\to\infty}\lambda_1(\rho)=+\infty$. Then

$$\sum_{i=0}^{\infty}\#\left\{(k,l):k\in\mathbb{Z}^+,l\in\mathbb{Z}^+\cup\{0\},\left(\frac{\pi l}{\delta}\right)^2+\lambda_k(i\delta)<\lambda\right\}$$
$$\leqslant(1+\lambda^{1/2}\delta\pi^{-1})$$
$$\cdot\#\{k:k\in\mathbb{Z}^+,\lambda_k(0)<\lambda\}\cdot\#\{i:i\in\mathbb{Z}^+\cup\{0\},\lambda_1(i\delta)<\lambda\}<\infty. \tag{23}$$

By Dirichlet–Neumann bracketing [5, Proposition XI.2.4]

$$N_{D^+}(\lambda)\leqslant\sum_{i=0}^{\infty}\#\left\{(k,l):k\in\mathbb{Z}^+,l\in\mathbb{Z}^+\cup\{0\},\left(\frac{\pi l}{\delta}\right)^2+\lambda_k(i\delta)<\lambda\right\}, \tag{24}$$

and $N_D(\lambda)\leqslant N_{D^+}(\lambda)$.

Let $D^-\subseteq\mathbb{R}^m$ be the (one-sided) horn-shaped set given by

$$D^-(\rho)=\begin{cases}D(\delta(1+[\rho/\delta])), & \rho>0,\\ \phi, & \rho\leqslant 0,\end{cases} \tag{25}$$

and let

$$D_0^- = D^- \Big\backslash \bigcup_{n=1}^{\infty} (P_{n\delta} \cap D^-). \tag{26}$$

The spectrum of the Dirichlet laplacian $-\Delta_{D_0^-}$ is given by

$$\bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{l=1}^{\infty} \left\{ \left(\frac{\pi l}{\delta}\right)^2 + \lambda_k(i\delta) \right\}. \tag{27}$$

Suppose $N_D(\lambda) < \infty$. Since $D \supseteq D^- \supseteq D_0^-$ we have

$$\begin{aligned} N_D(\lambda) &\geqslant N_{D^-}(\lambda) \geqslant N_{D_0^-}(\lambda) \\ &= \sum_{i=1}^{\infty} \# \left\{ (k, l) : k \in \mathbb{Z}^+, l \in \mathbb{Z}^+, \left(\frac{\pi l}{\delta}\right)^2 + \lambda_k(i\delta) < \lambda \right\} \\ &\geqslant \# \left\{ i : \lambda_1(i\delta) < \lambda - \left(\frac{\pi}{\delta}\right)^2 \right\}. \end{aligned} \tag{28}$$

It follows that $\lim_{i \to \infty} \lambda_1(i\delta) = +\infty$. Since $\lambda_1(\rho)$ is monotone in ρ we conclude $\lim_{\rho \to \infty} \lambda_1(\rho) = +\infty$.

THEOREM 8. *Let D be a one-sided horn-shaped set in $\mathbb{R}^m$. Suppose that the spectrum of $-\Delta_{D_0}$ is discrete and that $\lim_{\rho \to \infty} \lambda_1(\rho) = +\infty$. Then for $\delta > 0$ and $\lambda > 0$*

$$\begin{aligned} &\left| N_D(\lambda) - \pi^{-1} \int_{[0,\infty)} dx \sum_{k=1}^{\infty} \{\{\lambda - \lambda_k(x)\}^+\}^{1/2} \right| \\ &\qquad \leqslant (1 + \delta\lambda^{1/2}\pi^{-1}) \cdot \# \{k : k \in \mathbb{Z}^+, \lambda_k(0) < \lambda\} \\ &\qquad\quad + \sum_{i=1}^{\infty} \# \{k : k \in \mathbb{Z}^+, \lambda_k(i\delta) < \lambda\}. \end{aligned} \tag{29}$$

Proof. By (19) we have

$$\begin{aligned} N_D(\lambda) &\leqslant \sum_{i=0}^{\infty} \# \{(k, l) : k \in \mathbb{Z}^+, l \in \mathbb{Z}^+ \cup \{0\}, (\pi l/\delta)^2 + \lambda_k(i\delta) < \lambda\} \\ &= \# \{(k, l) : k \in \mathbb{Z}^+, l \in \mathbb{Z}^+ \cup \{0\}, (\pi l/\delta)^2 + \lambda_k(0) < \lambda\} \\ &\quad + \sum_{i=1}^{\infty} \# \{k : k \in \mathbb{Z}^+, \lambda_k(i\delta) < \lambda\} \\ &\quad + \sum_{i=1}^{\infty} \# \{(k, l) : k \in \mathbb{Z}^+, l \in \mathbb{Z}^+, (\pi l/\delta)^2 + \lambda_k(i\delta) < \lambda\} \end{aligned}$$

$$
\begin{aligned}
&= (1+\delta\lambda^{1/2}\pi^{-1})\cdot \#\ \{k : k\in\mathbb{Z}^+, \lambda_k(0)<\lambda\} \\
&\quad + \sum_{i=1}^{\infty} \#\ \{k : k\in\mathbb{Z}^+, \lambda_k(i\delta)<\lambda\} \\
&\quad + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} [\delta\pi^{-1}\{\{\lambda-\lambda_k(i\delta)\}^+\}^{1/2}] \\
&\leqslant (1+\delta\lambda^{1/2}\pi^{-1})\cdot \#\ \{k : k\in\mathbb{Z}^+, \lambda_k(0)<\lambda\} \\
&\quad + \sum_{i=1}^{\infty} \#\ \{k : k\in\mathbb{Z}^+, \lambda_k(i\delta)<\lambda\} \\
&\quad + \delta\pi^{-1}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \{\{\lambda-\lambda_k(i\delta)\}^+\}^{1/2} \\
&\leqslant (1+\delta\lambda^{1/2}\pi^{-1})\cdot \#\ \{k : k\in\mathbb{Z}^+, \lambda_k(0)<\lambda\} \\
&\quad + \sum_{i=1}^{\infty} \#\ \{k : k\in\mathbb{Z}^+, \lambda_k(i\delta)<\lambda\} \\
&\quad + \pi^{-1}\int_{[0,\infty)} dx \sum_{k=1}^{\infty} \{\{\lambda-\lambda_k(x)\}^+\}^{1/2}, \qquad (30)
\end{aligned}
$$

since $x \to \lambda_k(x)$ is non-decreasing in x. Furthermore

$$
\begin{aligned}
N_D(\lambda) &\geqslant \sum_{i=1}^{\infty} \#\ \{(k,l) : k\in\mathbb{Z}^+, l\in\mathbb{Z}^+, (\pi l/\delta)^2+\lambda_k(i\delta)<\lambda\} \\
&= \sum_{i=1}^{\infty}\ \sum_{\{k : k\in\mathbb{Z}^+, \lambda_k(i\delta)<\lambda\}} [\delta\pi^{-1}\{\lambda-\lambda_k(i\delta)\}^{1/2}] \\
&\geqslant \sum_{i=1}^{\infty}\ \sum_{\{k : k\in\mathbb{Z}^+, \lambda_k(i\delta)<\lambda\}} \{\delta\pi^{-1}\{\lambda-\lambda_k(i\delta)\}^{1/2}-1\} \\
&= \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \delta\pi^{-1}\{\{\lambda-\lambda_k(i\delta)\}^+\}^{1/2} \\
&\quad - \sum_{i=1}^{\infty} \#\ \{k : k\in\mathbb{Z}^+, \lambda_k(i\delta)<\lambda\} \\
&\quad - \sum_{k=1}^{\infty} \delta\pi^{-1}\{\{\lambda-\lambda_k(0)\}^+\}^{1/2} \\
&\geqslant \pi^{-1}\int_{[0,\infty)} dx \sum_{k=1}^{\infty} \{\{\lambda-\lambda_k(x)\}^+\}^{1/2} \\
&\quad - \sum_{i=1}^{\infty} \#\ \{k : k\in\mathbb{Z}^+, \lambda_k(i\delta)<\lambda\} \\
&\quad - \delta\lambda^{1/2}\pi^{-1}\cdot \#\ \{k : k\in\mathbb{Z}^+, \lambda_k(0)<\lambda\}. \qquad (31)
\end{aligned}
$$

Proof of Theorem 2. Let D be as in Theorem 2. Then D satisfies the conditions in Theorem 8. Furthermore for $x \geqslant 0$ and $k = 1, 2, \ldots,$

$$\lambda_k(x) = (\pi k/f(x))^2. \tag{32}$$

Hence

$$\# \{k : k \in \mathbb{Z}^+, \lambda_k(0) < \lambda\} \leqslant \lambda^{1/2} f(0)\, \pi^{-1}, \tag{33}$$

and

$$\begin{aligned} &\sum_{i=1}^{\infty} \# \{k : k \in \mathbb{Z}^+, \lambda_k(i\delta) < \lambda\} \\ &\quad = \sum_{i=1}^{\infty} [f(i\delta)\, \lambda^{1/2} \pi^{-1}] \leqslant \sum_{\{i \in \mathbb{Z}^+ : f(i\delta)\, \lambda^{1/2} \geqslant \pi\}} f(i\delta)\, \lambda^{1/2} \pi^{-1} \\ &\quad \leqslant \lambda^{1/2} (\delta\pi)^{-1} \int_{\{x : f(x)\, \lambda^{1/2} \geqslant \pi\}} dx\, f(x), \end{aligned} \tag{34}$$

since $x \to f(x)$ is non-increasing. Choose

$$\delta = \pi^{1/2} \lambda^{-1/4} (f(0))^{-1/2} \left\{ \int_{\{x : f(x)\, \lambda^{1/2} \geqslant \pi\}} dx\, f(x) \right\}^{1/2}. \tag{35}$$

Theorem 2 follows from (29), (33), (34), and (35).

3. Proofs of the Corollaries

Proof of Corollary 5. By taking the Laplace transform of (10) we obtain by Cauchy–Schwarz

$$\begin{aligned} &\left| \int_{[0, \infty)} e^{-t\lambda} N_D(\lambda)\, d\lambda - \int_{[0, \infty)} d\lambda\, e^{-t\lambda} \int_{[0, \infty)} dx \sum_{k=1}^{\infty} \left\{ \left\{ \frac{\lambda}{\pi^2} - \frac{k^2}{f^2(x)} \right\}^+ \right\}^{1/2} \right| \\ &\quad \leqslant \int_{[0, \infty)} e^{-t\lambda} f(0)\, \lambda^{1/2} \pi^{-1}\, d\lambda \\ &\qquad + 2(f(0))^{1/2}\, \pi^{-3/2} \int_{[0, \infty)} d\lambda\, e^{-t\lambda} \lambda^{3/4} \left\{ \int_{\{x : f(x)\, \lambda^{1/2} \geqslant \pi\}} dx\, f(x) \right\}^{1/2} \\ &\quad \leqslant f(0)\, 4^{-1} \pi^{-1/2} t^{-3/2} \\ &\qquad + 2(f(0))^{1/2}\, \pi^{-3/2} \left\{ \int_{[0, \infty)} e^{-t\lambda} \lambda^{3/2}\, d\lambda \right\}^{1/2} \\ &\qquad \cdot \left\{ \int_{[0, \infty)} e^{-t\lambda}\, d\lambda \int_{\{x : f(x) \geqslant \pi\lambda^{-1/2}\}} dx\, f(x) \right\}^{1/2} \\ &\quad = f(0)\, 4^{-1} \pi^{-1/2} t^{-3/2} \\ &\qquad + (3f(0))^{1/2}\, \pi^{-5/4} t^{-7/4} \left\{ \int_{[0, \infty)} dx\, f(x)\, e^{-t\pi^2 f^{-2}(x)} \right\}^{1/2}. \end{aligned} \tag{36}$$

An integration by parts yields

$$\left| Z_D(t) - (4\pi t)^{-1/2} \int_{[0,\infty)} dx \sum_{k=1}^{\infty} e^{-t\pi^2 k^2 f^{-2}(x)} \right|$$
$$\leqslant f(0)\, t^{-1/2} + (f(0))^{1/2}\, t^{-3/4} \left\{ \int_{[0,\infty)} dx\, f(x)\, e^{-t\pi^2 f^{-2}(x)} \right\}^{1/2}. \quad (37)$$

Finally

$$(4\pi t)^{-1/2} \int_{[0,\infty)} dx \sum_{k=1}^{\infty} e^{-t\pi^2 k^2 f^{-2}(x)}$$
$$\geqslant (4\pi t)^{-1/2} \int_{[0,\infty)} dx \sum_{k=0}^{\infty} e^{-t\pi^2 f^{-2}(x)(1+3k^2)}$$
$$\geqslant (4\pi t)^{-1/2} \int_{[0,\infty)} dx \int_{[0,\infty)} dk\, e^{-t\pi^2 f^{-2}(x)(1+3k^2)}$$
$$= (4\pi t)^{-1}\, 3^{-1/2} \int_{[0,\infty)} dx\, f(x)\, e^{-t\pi^2 f^{-2}(x)}$$
$$\geqslant (4\pi t)^{-1}\, 3^{-1/2} (f(1))^{1/2}\, e^{-t\pi^2/(2f^2(1))} \left\{ \int_{[0,\infty)} dx\, f(x)\, e^{-t\pi^2 f^{-2}(x)} \right\}^{1/2}. \quad (38)$$

Corollary 5 follows from (37) and (38).

Proof of Corollary 3. Consider the case $\alpha > 1$. To obtain a lower bound respectively upper bound for N_{D_α} we put Dirichlet conditions, respectively Neumann conditions, on the edges of the square S with vertices $(1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$. Then D_α is partitioned into S and four horn-shaped regions H_1, H_2, H_3, H_4 such that $(2, 0) \in H_1$, $(0, -2) \in H_2$, $(-2, 0) \in H_3$, and $(0, 2) \in H_4$. Then for either Neumann or Dirichlet conditions on ∂S

$$N_S(\lambda) = \frac{\lambda}{\pi} + O(\lambda^{1/2}). \quad (39)$$

For either Neumann or Dirichlet conditions on $\partial H_1 \cap \partial S$, respectively $\partial H_3 \cap \partial S$, one has by Theorem 2 with $f(x) = 2(1+x)^{-1/\alpha}$

$$N_{H_1}(\lambda) + N_{H_3}(\lambda)$$
$$= 2 \int_{[0,\infty)} dx \sum_{k=1}^{\infty} \left\{ \left\{ \frac{\lambda}{\pi^2} - 4^{-1} k^2 (1+x)^{2/\alpha} \right\}^{+} \right\}^{1/2} + O(\lambda^{(2+\alpha)/4}). \quad (40)$$

Moreover

$$2\int_{[0,\infty)} dx \sum_{k=1}^{\infty} \left\{\left\{\frac{\lambda}{\pi^2} - 4^{-1}k^2(1+x)^{2/\alpha}\right\}^+\right\}^{1/2}$$
$$= \beta(\alpha)\,\lambda^{(1+\alpha)/2} - 2\int_{[0,1)} dx \sum_{k=1}^{\infty} \left\{\left\{\frac{\lambda}{\pi^2} - 4^{-1}k^2x^{2/\alpha}\right\}^+\right\}^{1/2}$$
$$= \beta(\alpha)\,\lambda^{(1+\alpha)/2} - \frac{\lambda\alpha}{\pi(\alpha-1)} + O(\lambda^{1/2}). \tag{41}$$

Similarly for either Neumann or Dirichlet conditions on $\partial H_2 \cap \partial S$, respectively $\partial H_4 \cap \partial S$, one has by Theorem 2 with $f(x) = 2(1+x)^{-\alpha}$

$$N_{H_2}(\lambda) + N_{H_4}(\lambda) = 2\int_{[0,\infty)} dx \sum_{k=1}^{\infty} \left\{\left\{\frac{\lambda}{\pi^2} - 4^{-1}k^2(1+x)^{2\alpha}\right\}^+\right\}^{1/2} + O(\lambda^{3/4}). \tag{42}$$

Moreover

$$2\int_{[0,\infty)} dx \sum_{k=1}^{\infty} \left\{\left\{\frac{\lambda}{\pi^2} - 4^{-1}k^2(1+x)^{2\alpha}\right\}^+\right\}^{1/2}$$
$$= \frac{\lambda}{\pi(\alpha-1)} - 2\int_{[0,1)} dx \sum_{k=1}^{\infty} \left\{\left\{\frac{\lambda}{\pi^2} - 4^{-1}k^2x^{2\alpha}\right\}^+\right\}^{1/2}$$
$$= \frac{\lambda}{\pi(\alpha-1)} + \beta(1/\alpha)\,\lambda^{(1+\alpha)/(2\alpha)} + O(\lambda^{1/2}). \tag{43}$$

Note that $\lambda/(\pi(\alpha-1))$ is the standard Weyl term, since H_2 and H_4 have finite measure. From (39)–(43) we obtain for $\alpha > 1$

$$N_{D_\alpha}(\lambda) = \beta(\alpha)\,\lambda^{(1+\alpha)/2} + \beta(1/\alpha)\,\lambda^{(1+\alpha)/(2\alpha)} + O(\lambda^{(2+\alpha)/4}). \tag{44}$$

Finally note that $(1+\alpha)/(2\alpha) > (2+\alpha)/4$ for $1 < \alpha < 2^{1/2}$. This proves (14) and (15). The proof of (13) is similar.

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